11. Homology

A complex is a particular type of partially ordered set with complementary properties designed to carry an algebraic superstructure, its homology theory. Complexes thus appear as the tool par excellence for the application of algebraic methods to topology.

Solomon Lefschetz, 1942

Simplicial complexes enjoy good topological properties. Their combinatorial structure is sufficiently rich via subdivision to capture the continuous mappings between realizations of complexes up to homotopy. In Chapter 10 we developed these connections between the combinatorial and the continuous. In this chapter we develop the combinatorial structure further by defining algebraic structures associated to a complex that will be found to give topological invariants. These invariants lead to a proof of the topological Invariance of Dimension which is a generalization of the argument in Chapter 8 in which the fundamental group played the key role for the case $(2, n)$.

The algebraic structures will be finite dimensional vector spaces over the field with two elements, $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$. Let's set some notation: If S is any finite set, then $\mathbb{F}_2[S]$ denotes the vector space over $\mathbb{F}_2$ with S as basis, that is, the set of all formal sums $\sum_{s\in S} a_s s$ where $a_s \in \mathbb{F}_2$. The sum of two such formal sums is given by

$$\sum_{s\in S} a_s s + \sum_{s\in S} b_s s = \sum_{s\in S}(a_s + b_s)s.$$

Multiplication by a scalar $c \in \mathbb{F}_2$ is given by $c\sum_{s\in S} a_s s = \sum_{s\in S} c a_s s$. The reader can check that these operations make $\mathbb{F}_2[S]$ a vector space. If S and T are finite sets, and $f: S \to \mathbb{F}_2[T]$ is a function, then f induces a linear mapping $f_*: \mathbb{F}_2[S] \to \mathbb{F}_2[T]$, given by

$$f_*\left(\sum_{s\in S} a_s s\right) = \sum_{s\in S} a_s f(s).$$

Since a linear mapping is determined by its values on a basis of the domain, this construction gives every linear mapping between $\mathbb{F}_2[S]$ and $\mathbb{F}_2[T]$.

The quotient construction of a vector space by a linear subspace (Chapter 1) will come up later, and we recall it here. Suppose W is a linear subspace of a vector space V. The **quotient vector space** V/W is the set of equivalence classes of vectors in V under the equivalence relation $v \sim v'$ if $v' - v \in W$. We denote the equivalence class of $v \in V$ by $[v]$ or $v + W$. The addition and multiplication by a scalar on V/W are given by $(v + W) + (v' + W) = (v + v') + W$ and $c(v + W) = cv + W$. When V is finite-dimensional, $\dim V/W = \dim V - \dim W$.

In Chapter 9 we associated to a grating $\mathcal{G}$ the vector space of i-chains, $C_i(\mathcal{G}) = \mathbb{F}_2[E_i(\mathcal{G})]$. We can generalize that construction to a simplicial complex: Suppose K is a simplicial complex (geometric or abstract). Partition K into disjoint subsets that contain only nondegenerate simplices of a fixed dimension:

$$K_p = \{S \in K \mid \dim S = p \text{ and } S \text{ is nondegenerate}\}.$$

The index p varies from zero to the dimension of K. Each K_p is a finite set which forms the basis for the **p-chains** on K,

$$C_p(K;\mathbb{F}_2) = \mathbb{F}_2[K_p], \text{ the vector space over } \mathbb{F}_2 \text{ with basis } K_p.$$

A typical element of $C_p(K;\mathbb{F}_2)$ is a sum $S_1+S_2+\cdots+S_l$, where each S_i is a nondegenerate p-simplex in K. When working over $\mathbb{F}_2$, recall that $S+S = 2\cdot S = 0\cdot S = \mathbf{0}$ in $C_p(K;\mathbb{F}_2)$.

A simplicial mapping $\phi\colon K \to L$ induces a linear mapping $\phi_*\colon C_p(K;\mathbb{F}_2) \to C_p(L;\mathbb{F}_2)$ defined on a p-simplex $S = \{v_0,\ldots,v_p\}$ by

$$\phi_*(\{v_0,\ldots,v_p\}) = \begin{cases} \{\phi(v_0),\ldots,\phi(v_p)\} & \text{if } \{\phi(v_0),\ldots,\phi(v_p)\} \text{ is nondegenerate in } L, \\ \mathbf{0} & \text{if } \{\phi(v_0),\ldots,\phi(v_p)\} \text{ is degenerate in } L, \end{cases}$$

and defined on a chain $c = S_1+\cdots+S_l$ by

$$\phi_*(c) = \phi_*(S_1+\cdots+S_l) = \phi_*(S_1)+\cdots+\phi_*(S_l).$$

If we have two simplicial mappings $\phi\colon K \to L$ and $\psi\colon L \to M$, then the composite $\psi\circ\phi\colon K \to M$ induces a mapping $(\psi\circ\phi)_*\colon C_p(K;\mathbb{F}_2) \to C_p(M;\mathbb{F}_2)$ which satisfies the equation $(\psi\circ\phi)_* = \psi_*\circ\phi_*$.

In Chapter 10 we introduced the face of a p-simplex $S = \{v_0,\ldots,v_p\}$ opposite a vertex v_i, given by the subset $\partial_i(S) = \{v_0,\ldots,\widehat{v_i},\ldots,v_p\} \subset S$ (the vertex under the hat is omitted). Notice that if S is nondegenerate, then so is $\partial_i S$. Define a mapping $\partial\colon K_p \to C_{p-1}(K;\mathbb{F}_2)$ by summing all of the $(p-1)$-faces of a p-simplex. The extension of ∂ to a linear mapping $C_p(K;\mathbb{F}_2) \to C_{p-1}(K;\mathbb{F}_2)$ is called the **boundary homomorphism**:

$$\partial\colon C_p(K;\mathbb{F}_2) \to C_{p-1}(K;\mathbb{F}_2) \text{ given by } \partial(S) = \sum\nolimits_{i=0}^{p} \partial_i(S), \text{ for } S \in K_p.$$

Recall from Chapter 10 that $\operatorname{bdy}\Delta^n[S] = \bigcup_{i=0}^{p} \Delta^{n-1}[\partial_i(S)]$. The boundary homomorphism ∂ is an algebraic version of bdy, the topological boundary operation.

The main algebraic properties of the boundary homomorphism are the following:

PROPOSITION 11.1. *If $\phi\colon K \to L$ is a simplicial mapping, then*

$$\partial\circ\phi_* = \phi_*\circ\partial\colon C_p(K;\mathbb{F}_2) \to C_{p-1}(L;\mathbb{F}_2).$$

Furthermore, the composite $\partial\circ\partial\colon C_p(K;\mathbb{F}_2) \to C_{p-2}(K;\mathbb{F}_2)$ is the zero mapping.

Proof: It suffices to check these equations for elements in a basis. Suppose that $S = \{v_0,\ldots,v_p\}$ is a nondegenerate p-simplex in K. Then

$$\begin{aligned} \partial\circ\phi_*(S) &= \partial(\{\phi(v_0),\ldots,\phi(v_p)\}) = \sum\nolimits_{i=0}^{p} \{\phi(v_0),\ldots,\widehat{\phi(v_i)},\ldots,\phi(v_p)\} \\ &= \sum\nolimits_{i=0}^{p} \phi_*(\{v_0,\ldots,\widehat{v_i},\ldots,v_p\}) = \phi_*\left(\sum\nolimits_{i=0}^{p}\{v_0,\ldots,\widehat{v_i},\ldots,v_p\}\right) \\ &= \phi_*\circ\partial(S). \end{aligned}$$

Next, we compute $\partial \circ \partial(S)$.

$$\begin{aligned}\partial(\partial(S)) &= \partial\left(\sum\nolimits_{i=0}^{p}\{v_0,\ldots,\widehat{v_i},\ldots,v_p\}\right)\\ &= \sum\nolimits_{j<i}\sum\nolimits_{i=0}^{p}\{v_0,\ldots,\widehat{v_j},\ldots,\widehat{v_i},\ldots,v_p\} + \sum\nolimits_{j>i}\sum\nolimits_{i=0}^{p}\{v_0,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_p\}.\end{aligned}$$

Notice, for each pair $k < l$, the $(p-2)$-simplex $\{v_0,\ldots,\widehat{v_k},\ldots,\widehat{v_l},\ldots,v_p\}$ appears twice, once in each sum, and so $\partial(\partial(S)) = \mathbf{0}$. $\diamondsuit$

The boundary homomorphism determines certain linear subspaces of $C_p(K;\mathbb{F}_2)$: the **space of p-cycles**,

$$Z_p(K) = \ker(\partial\colon C_p(K;\mathbb{F}_2) \to C_{p-1}(K;\mathbb{F}_2)) = \{c \in C_p(K;\mathbb{F}_2) \mid \partial(c) = \mathbf{0}\},$$

and the **space of p-boundaries**,

$$\begin{aligned}B_p(K) &= \partial(C_{p+1}(K;\mathbb{F}_2)) = \mathrm{im}\ (\partial\colon C_{p+1}(K;\mathbb{F}_2) \to C_p(K;\mathbb{F}_2))\\ &= \{b \in C_p(K;\mathbb{F}_2) \mid b = \partial(c),\ \text{for some } c \in C_{p+1}(K;\mathbb{F}_2)\}.\end{aligned}$$

The relation $\partial \circ \partial = 0$ implies the inclusion $B_p(K) \subset Z_p(K)$.

For a p-simplex S, the boundary $\partial(S)$ is a cycle that is the sum of the faces $\partial_i(S)$ and together these make up the boundary of $\Delta^p[S]$. When faces come together like this, but the simplex whose boundary they form is absent, we get a 'p-dimensional hole' in the realization of the simplicial complex. The vector space of the essential cycles—holes not filled in as the boundary of a higher dimensional simplex—is algebraically expressed as the quotient vector space $Z_p(K)/B_p(K)$. This is the homology in dimension p of a simplicial complex.

Definition 11.2. *The pth* **homology** *(mod 2) of a simplicial complex K is the quotient vector space for $p > 0$ given by*

$$H_p(K;\mathbb{F}_2) = Z_p(K)/B_p(K).$$

When $p = 0$, define $H_0(K;\mathbb{F}_2) = C_0(K;\mathbb{F}_2)/B_0(K)$.

To illustrate the definition, we compute the homology of the one-point complex, $\mathbf{\Delta}^0 = \{v\}$. In this case, the 0-chains have a single vertex $\{v\}$ for a basis, and the boundary homomorphism is zero. Since there are no other simplices, $H_0(\mathbf{\Delta}^0;\mathbb{F}_2) = \mathbb{F}_2[\{v\}]$, and $H_p(\mathbf{\Delta}^0;\mathbb{F}_2) = \{\mathbf{0}\}$ for $p > 0$.

A slightly more complicated computation is the homology of a 1-simplex, $\mathbf{\Delta}^1 \cong \Delta^1[S]$ where $S = \{\mathbf{e}_0, \mathbf{e}_1\}$: the chains and boundary homomorphisms may be assembled into a sequence of vector spaces and linear mappings:

$$\{\mathbf{0}\} \to C_1(\mathbf{\Delta}^1;\mathbb{F}_2) \xrightarrow{\partial} C_0(\mathbf{\Delta}^1;\mathbb{F}_2) \to \{\mathbf{0}\} \iff \{\mathbf{0}\} \to \mathbb{F}_2[\{S\}] \xrightarrow{\partial} \mathbb{F}_2[\{\mathbf{e}_0,\mathbf{e}_1\}] \to \{\mathbf{0}\}.$$

Since $\partial(S) = \mathbf{e}_0 + \mathbf{e}_1 \neq \mathbf{0}$, there is no kernel in dimension one, and the zero boundaries are given by $B_0(\mathbf{\Delta^1}) = \mathbb{F}_2[\{\mathbf{e}_0 + \mathbf{e}_1\}]$. Thus $H_0(\mathbf{\Delta^1};\mathbb{F}_\mathbf{2}) \cong \mathbb{F}_\mathbf{2}[\{[\mathbf{e_0}]\}]$ where the equivalence class $[\mathbf{e}_0] = \mathbf{e}_0 + \mathbb{F}_2[\{\mathbf{e}_0 + \mathbf{e}_1\}]$ is the coset of $\mathbf{e}_0$ in the quotient $\mathbb{F}_2[\{\mathbf{e}_0,\mathbf{e}_1\}]/\mathbb{F}_2[\{\mathbf{e}_0 + \mathbf{e}_1\}]$.

To generalize this computation to $H_p(\mathbf{\Delta^n}; \mathbb{F}_\mathbf{2})$ for all n and p, we introduce a linear mapping fashioned from the combinatorics of a simplex. Let $S = \{v_0, \ldots, v_n\}$ denote a nondegenerate n-simplex. Consider the linear mapping $i_{v_n}\colon C_p(\Delta^n[S]; \mathbb{F}_2) \to C_{p+1}(\Delta^n[S]; \mathbb{F}_2)$ given on the basis by

$$i_{v_n}(\{v_{i_0}, \ldots, v_{i_p}\}) = \begin{cases} \{v_{i_0}, \ldots, v_{i_p}, v_n\} & \text{if } \{v_{i_0}, \ldots, v_{i_p}, v_n\} \text{ is nondegenerate,} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

If $\{v_{i_0}, \ldots, v_{i_p}\}$ is a nondegenerate p-simplex in $\Delta^n[S]$, $p > 0$, and $v_n \neq v_{i_k}$ for all k, we can compute

$$\begin{aligned} &(\partial \circ i_{v_n} + i_{v_n} \circ \partial)(\{v_{i_0}, \ldots, v_{i_p}\}) \\ &\qquad = \partial(\{v_{i_0}, \ldots, v_{i_p}, v_n\}) + i_{v_n}\left(\sum_{r=0}^{p} \{v_{i_0}, \ldots, \widehat{v_{i_r}}, \ldots, v_{i_p}\}\right) \\ &\qquad = \{v_{i_0}, \ldots, v_{i_p}\} + \sum_{r=0}^{p} \{v_{i_0}, \ldots, \widehat{v_{i_r}}, \ldots, v_{i_p}, v_n\} + \sum_{r=0}^{p} \{v_{i_0}, \ldots, \widehat{v_{i_r}}, \ldots, v_{i_p}, v_n\} \\ &\qquad = \{v_{i_0}, \ldots, v_{i_p}\}. \end{aligned}$$

When $S = \{v_{i_0}, \ldots, v_{i_{p-1}}, v_n\}$, then $(\partial \circ i_{v_n} + i_{v_n} \circ \partial)(S) = S + U$, where U is a sum of degenerate $(p+1)$-simplices which we take to be $\mathbf{0} \in C_{p+1}(K; \mathbb{F}_2)$. It follows that $\partial \circ i_{v_n} + i_{v_n} \circ \partial = \mathrm{id}$, and if z is a p-cycle, then

$$z = (\partial \circ i_{v_n} + i_{v_n} \circ \partial)(z) = \partial(i_{v_n}(z)) \in B_p(K).$$

Hence, for $p > 0$, $Z_p(K) \subset B_p(K) \subset Z_p(K)$ and so $H_p(\Delta^n[S]; \mathbb{F}_2) = \{\mathbf{0}\}$.

To compute $H_0(\Delta^n[S]; \mathbb{F}_2)$, notice that $\partial(i_{v_n}(v)) = v + v_n$ while $i_{v_n}(\partial(v)) = \mathbf{0}$. The equation $\partial \circ i_{v_n} + i_{v_n} \circ \partial = \mathrm{id}$ does not hold, but we can deduce that $v_n + B_0(\Delta^n[S]) = v_i + B_0(\Delta^n[S])$ for all i. Since $Z_0(\Delta^n[S]) = C_0(\Delta^n[S]; \mathbb{F}_2) = \mathbb{F}_2[\{v_0, \ldots, v_q\}]$, we have

$$H_0(\Delta^n[S]; \mathbb{F}_2) \cong C_0(\Delta^n[S]; \mathbb{F}_2)/\mathbb{F}_2[\{v + v' \mid v \neq v', v, v' \in S\}] \cong \mathbb{F}_2[\{v_n + B_0(\Delta^n[S])\}].$$

Notice that the homology of an n-simplex is isomorphic to the homology of a 0-simplex for all n.

We collect the vector spaces of p-chains on $\mathbf{\Delta}^n$ for all p, together with the boundary homomorphisms, to get a sequence of linear mappings

$$\{\mathbf{0}\} \to C_n(\mathbf{\Delta}^n; \mathbb{F}_2) \xrightarrow{\partial} C_{n-1}(\mathbf{\Delta}^n; \mathbb{F}_2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(\mathbf{\Delta}^n; \mathbb{F}_2) \xrightarrow{\partial} C_0(\mathbf{\Delta}^n; \mathbb{F}_2) \to \{\mathbf{0}\}.$$

From the formula $\partial \circ i_{v_n} + i_{v_n} \circ \partial = \mathrm{id}$, we found that, for $p > 0$, $Z_p(\mathbf{\Delta}^n) = B_p(\mathbf{\Delta}^n)$. In general, we say that a sequence of linear mappings $V \xrightarrow{a} W \xrightarrow{b} U$ is **exact at** W if $\ker b = \mathrm{im}\ a$. In the case of the sequence of chains on $\mathbf{\Delta}^n$, it is exact at $C_i(\mathbf{\Delta}^n; \mathbb{F}_2)$ for $1 \leq i \leq n$. In fact, the pth homology of a simplicial complex, $H_p(K; \mathbb{F}_2) = Z_p(K)/B_p(K)$, measures the failure of the sequence of boundary homomorphisms to be exact at $C_p(K; \mathbb{F}_2)$.

The exactness of the sequence of chains on $\mathbf{\Delta}^n$ gives a method for the computation of $H_p(\text{bdy}\,\mathbf{\Delta}^n;\mathbb{F}_2)$. The set of simplices of $\text{bdy}\,\mathbf{\Delta}^n$ contains all of the simplices of $\mathbf{\Delta}^n$ *except* the n-simplex $\{\mathbf{e}_0,\ldots,\mathbf{e}_n\}$. We can present the sequence of vector spaces of chains and boundary homomorphisms for $\text{bdy}\,\mathbf{\Delta}^n$ as

$$\{\mathbf{0}\} \to C_{n-1}(\mathbf{\Delta}^n;\mathbb{F}_2) \xrightarrow{\partial} C_{n-2}(\mathbf{\Delta}^n;\mathbb{F}_2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(\mathbf{\Delta}^n;\mathbb{F}_2) \xrightarrow{\partial} C_0(\mathbf{\Delta}^n;\mathbb{F}_2) \to \{\mathbf{0}\}.$$

We know that the sequence is exact at $C_i(\mathbf{\Delta}^n;\mathbb{F}_2)$ for $1 \leq i \leq n-2$, that the sequence used to be exact at $C_{n-1}(\mathbf{\Delta}^n;\mathbb{F}_2)$ and that $C_n(\mathbf{\Delta}^n;\mathbb{F}_2) = \mathbb{F}_2[\{\mathbf{e}_0,\ldots,\mathbf{e}_n\}]$. In the sequence for $\text{bdy}\,\mathbf{\Delta}^n$, the vector space of $(n-1)$-cycles $Z_{n-1}(\text{bdy}\,\mathbf{\Delta}^n)$ has dimension one. Since $B_{n-1}(\text{bdy}\,\mathbf{\Delta}^n) = \{\mathbf{0}\}$, we deduce that

$$H_p(\text{bdy}\,\mathbf{\Delta}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } p=0 \text{ or } p=n-1, \\ \{\mathbf{0}\}, & \text{otherwise.} \end{cases}$$

As we showed in Chapter 10, the realization $|\text{bdy}\,\mathbf{\Delta}^n|$ is homeomorphic to S^{n-1}. Later we will show how the homology of $\text{bdy}\,\mathbf{\Delta}^n$ can be associated to the topological space S^{n-1}.

To a simplicial complex K we can associate a number based on the combinatorial data of the simplices: Recall the subsets $K_p \subset K$ given by the nondegenerate p-simplices of K. Since K is a finite set, K_p is finite. Let $n_p = \#K_p$, the cardinality of K_p. The **Euler-Poincaré characteristic** of K is the alternating sum

$$\chi(K) = \sum\nolimits_{p=0}^{d} (-1)^p n_p,$$

where d denotes the dimension of K. This number was introduced by Euler in 1750 in a letter to Christian Goldbach (1690-1764). Euler's formula, $v - e + f = 2$, applies to two-dimensional polyhedra that are homeomorphic to the sphere, but we are getting a little ahead of the story. Here $v = \#\,\text{vertices} = n_0$, $e = \#\,\text{edges} = n_1$ and $f = \#\,\text{faces} = n_2$. For example, for the tetrahedron, $\text{bdy}\,\mathbf{\Delta}^3$, we have $v=4$, $e=6$ and $f=4$.

An extraordinary property of $\chi(K)$ is that it is calculable from the homology.

Theorem 11.3. *If K is a simplicial complex with $\chi(K) = \sum_{p=0}^{d}(-1)^p n_p$, then $\chi(K) = \sum_{p=0}^{d}(-1)^p h_p$, where $h_p = \dim_{\mathbb{F}_2} H_p(K;\mathbb{F}_2)$.*

Proof: By definition $n_p = \#K_p = \dim_{\mathbb{F}_2} C_p(K;\mathbb{F}_2)$. There are other numbers associated to the chains via the boundary operator. Let

$$\begin{aligned} z_p &= \dim_{\mathbb{F}_2} \ker(\partial\colon C_p(K;\mathbb{F}_2) \to C_{p-1}(K;\mathbb{F}_2)), \\ b_p &= \dim_{\mathbb{F}_2} \text{im}\,(\partial\colon C_{p+1}(K;\mathbb{F}_2) \to C_p(K;\mathbb{F}_2)). \end{aligned}$$

By definition $h_p = \dim_{\mathbb{F}_2} H_p(K;\mathbb{F}_2) = \dim_{\mathbb{F}_2} Z_p(K)/B_p(K) = z_p - b_p$. The fundamental identity from linear algebra for linear mappings, that the dimension of the domain of a mapping is equal to the dimension of its kernel plus the dimension of its image, implies that $n_p = z_p + b_{p-1}$. Manipulating these identities, we have

$$\begin{aligned}
\chi(K) &= \sum\nolimits_{p=0}^{d}(-1)^p n_p = \sum\nolimits_{p=0}^{d}(-1)^p(z_p + b_{p-1}) \\
&= (-1)^d(z_d + b_{d-1}) + (-1)^{d-1}(z_{d-1} + b_{d-2}) + \cdots + (-1)(z_1 + b_0) + z_0 \\
&= (-1)^d z_d + (-1)^{d-1}(z_{d-1} - b_{d-1}) + \cdots + (-1)(z_1 - b_1) + (z_0 - b_0) \\
&= (-1)^d h_d + (-1)^{d-1} h_{d-1} + \cdots + (-1)h_1 + h_0 = \sum\nolimits_{p=0}^{d}(-1)^p h_p.
\end{aligned}$$

Thus, the number $\chi(K)$ is calculable from the homology of K. $\diamondsuit$

Poincaré generalized Euler's formula by this argument in [Poincare] an 1895 paper that established the importance of this circle of ideas.

Homology and simplicial mappings

Suppose $\phi: K \to L$ is a simplicial mapping. Then ϕ induces a linear mapping of chains, $\phi_*: C_p(K;\mathbb{F}_2) \to C_p(L;\mathbb{F}_2)$, for which $\partial \circ \phi_* = \phi_* \circ \partial$. Suppose $[c] = c + B_p(K)$ denotes an element in $H_p(K;\mathbb{F}_2)$. Then $c \in Z_p(K)$, that is, $\partial(c) = \mathbf{0}$, and $\partial(\phi_*(c)) = \phi_*(\partial(c)) = \mathbf{0}$, so $\phi_*(c)$ is an element of $Z_p(L)$. If $c - c' \in B_p(K)$, then $\phi_*(c - c') = \phi_*(\partial(u)) = \partial(\phi_*(u))$, for some $u \in C_{p+1}(K;\mathbb{F}_2)$, and so $\phi_*(c) + B_p(L) = \phi_*(c') + B_p(L)$. Thus we can define

$$H(\phi): H_p(K;\mathbb{F}_2) \to H_p(L;\mathbb{F}_2) \text{ by } H(\phi)(c + B_p(K)) = \phi_*(c) + B_p(L).$$

It follows from the properties of the induced mappings on chains that if $\psi: L \to M$ is another simplicial mapping, then $H(\psi \circ \phi) = H(\psi) \circ H(\phi)$. We note also that the identity mapping $\mathrm{id}: K \to K$ induces the identity mapping $H(\mathrm{id}) = \mathrm{id}: H_p(K;\mathbb{F}_2) \to H_p(K;\mathbb{F}_2)$ for all p.

Although there are only finitely many simplicial mappings $\phi: K \to L$, there can be other linear mappings $C_p(K;\mathbb{F}_2) \to C_q(L;\mathbb{F}_2)$, which, like i_{v_n}, are defined using the features of simplices which make up the bases. The following notion was introduced by Lefschetz [Lefschetz1930].

Definition 11.4. *Given two simplicial mappings ϕ and $\psi: K \to L$, there is a* **chain homotopy** *between them if there is a linear mapping $h: C_p(K;\mathbb{F}_2) \to C_{p+1}(L;\mathbb{F}_2)$ for each p which satisfies*

$$\partial \circ h + h \circ \partial = \phi_* + \psi_*.$$

Theorem 11.5. *If there is a chain homotopy between ϕ and ψ, then $H(\phi) = H(\psi)$.*

Proof: Suppose $[c] = c + B_p(K) \in H_p(K;\mathbb{F}_2)$. Then

$$\partial \circ h(c) + h \circ \partial(c) = \phi_*(c) + \psi_*(c).$$

Since $\partial(c) = \mathbf{0}$, $\phi_*(c) + \psi_*(c) = \partial(h(c)) \in B_p(L)$, that is, $\phi_*(c) + B_p(L) = \psi_*(c) + B_p(L)$ and $H(\phi)([c]) = H(\psi)([c])$. $\diamondsuit$

An important source of chain homotopies is the combinatorial notion of contiguous simplicial mappings. Recall that simplicial mappings $\phi, \psi: K \to L$ are *contiguous* if, for any simplex $S \in K$, we have $\phi(S) \cup \psi(S)$ is a simplex in L.

Corollary 11.6. *If ϕ and $\psi\colon K \to L$ are simplicial mappings, and ϕ is contiguous to ψ, then $H(\phi) = H(\psi)\colon H_p(K;\mathbb{F}_2) \to H_p(L;\mathbb{F}_2)$ for all p.*

Proof: Define the linear mapping $h\colon C_p(K;\mathbb{F}_2) \to C_{p+1}(L;\mathbb{F}_2)$ determined on the basis by

$$h(\{v_0,\ldots,v_p\}) = \sum\nolimits_{i=0}^{p} \{\phi(v_0),\ldots,\phi(v_i),\psi(v_i),\ldots,\psi(v_p)\},$$

where we substitute the zero element whenever we have a degenerate simplex in the sum. Since ϕ and ψ are contiguous, each summand of $h(\{v_0,\ldots,v_p\})$ is a simplex in L.

Then we can compute

$$\begin{aligned}
(\partial\circ h)(T) &= \partial(h(T)) = \partial\left(\sum\nolimits_{i=0}^{p}\{\phi(v_0),\ldots,\phi(v_i),\psi(v_i),\ldots,\psi(v_p)\}\right)\\
&= \sum\nolimits_{i=0}^{p}\sum\nolimits_{j\le i}\{\phi(v_0),\ldots,\widehat{\phi(v_j)},\ldots,\phi(v_i),\psi(v_i),\ldots,\psi(v_p)\}\\
&\quad+\sum\nolimits_{i=0}^{p}\sum\nolimits_{j\ge i}\{\phi(v_0),\ldots,\phi(v_i),\psi(v_i),\ldots,\widehat{\psi(v_j)},\ldots,\psi(v_p)\}\\
(h\circ\partial)(T) &= h(\partial(T)) = h\left(\sum\nolimits_{i=0}^{p}\{v_0,\ldots,\widehat{v_i},\ldots,v_p\}\right)\\
&= \sum\nolimits_{i=0}^{p}\sum\nolimits_{j<i}\{\phi(v_0),\ldots,\phi(v_j),\psi(v_j),\ldots,\widehat{\psi(v_i)},\ldots,\psi(v_p)\}\\
&\quad+\sum\nolimits_{i=0}^{p}\sum\nolimits_{j>i}\{\phi(v_0),\ldots,\widehat{\phi(v_i)},\ldots,\phi(v_j),\psi(v_j),\ldots,\psi(v_p)\}
\end{aligned}$$

The differences between these expressions are the inequalities $j < i$ and $j \le i$, and $j > i$ and $j \ge i$. In the sum for $\partial(h(T))$ the summands that do not appear in $h(\partial(T))$ are given by the condition $i = j$:

$$\sum\nolimits_{i=0}^{p}\{\phi(v_0),\ldots,\phi(v_{i-1}),\psi(v_i),\ldots,\psi(v_p)\} + \{\phi(v_0),\ldots,\phi(v_i),\psi(v_{i+1}),\ldots,\psi(v_p)\}.$$

Each entry appears twice in the sum, except when $i = 0$ and $i = p$, leaving

$$\{\phi(v_0),\ldots,\phi(v_p)\} + \{\psi(v_0),\ldots,\psi(v_p)\} = (\phi_* + \psi_*)(\{v_0,\ldots,v_p\}).$$

All of the summands in $h(\partial(T))$ are cancelled by the rest of the summands of $\partial(h(T))$ and so we have $\partial\circ h + h\circ\partial = \phi_* + \psi_*$, a chain homotopy between ϕ and ψ. By Theorem 11.5, $H(\phi) = H(\psi)$. $\diamondsuit$

By Lemma 10.19, Corollary 11.6 implies the following:

Corollary 11.7. *If ϕ and $\psi\colon K \to L$ are simplicial approximations of a continuous mapping $f\colon |K| \to |L|$, then $H(\phi) = H(\psi)\colon H_p(K;\mathbb{F}_2) \to H_p(L;\mathbb{F}_2)$, for all p.*

Since a single continuous mapping might have numerous simplicial approximations, when the domain and codomain are held fixed, the induced mappings on homology by these approximations are the same.

Topological invariance

So far we have associated a sequence of vector spaces over $\mathbb{F}_2$ to a simplicial complex. To fashion a tool for the investigation of topological questions, we need to associate homology vector spaces and linear mappings to spaces and continuous mappings. It would be nice to do this for general topological spaces, but it is not clear that it is possible to associate a finite simplicial complex to each space (it isn't [Spanier]). We restrict our attention to triangulable spaces, that is, spaces X for which there is a simplicial complex K with X homeomorphic to $|K|$. For such spaces it would be natural to define $H_p(X;\mathbb{F}_2) = H_p(K;\mathbb{F}_2)$. However, a triangulable space can be homeomorphic to many different simplicial complexes. For example, the sphere S^2 is homeomorphic to the tetrahedron, the octohedron, and the icosahedron. It is also the case (Thoerem 10.12) that we can subdivide a simplicial complex without changing its realization. How does homology behave under subdivision?

We also want to associate to a continuous mapping $f\colon X \to Y$, for each $p \geq 0$ a linear mapping $H(f)\colon H_p(X;\mathbb{F}_2) \to H_p(Y;\mathbb{F}_2)$. The natural guess is to take a simplicial approximation $\phi\colon \mathrm{sd}^N K \to L$ and define $H(f) = H(\phi)$. This definition is nearly well-defined because two simplicial approximations to the same mapping are contiguous. However, simplicial approximations to a single mapping can be constructed for which a different number of barycentric subdivisions might be needed, or a different choice of representing simplicial complexes might have been made and so it is not immediate that we have a good definition.

To alleviate some of the problems here, we loosen some of the foundations to allow a new precision. To allow different choices of a simplicial complex with realization homeomorphic to X we can define $H_p(X;\mathbb{F}_2)$ *up to isomorphism*, that is, do not associate a particular vector space to X and p, but an equivalence class of vector spaces in which a choice of simplicial complex determines a representative. The equivalence relation is isomorphism, that is, we say that vector spaces V and V' are equivalent if there is a linear isomorphism $\alpha\colon V \to V'$ between them. This relation on any set of vector spaces is reflexive, symmetric, and transitive. We also define a relation between linear mappings between equivalent vector spaces: if $\phi\colon V \to W$ and $\phi'\colon V' \to W'$ are linear mappings and V is isomorphic to V', W is isomorphic to W', then we say that ϕ is equivalent to ϕ' if there is a diagram of linear mappings

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\alpha'} \\ V' & \xrightarrow[\phi']{} & W' \end{array}$$

that is *commutative*, that is, $\alpha' \circ \phi = \phi' \circ \alpha$ and α and α' are isomorphisms. Once again, this relation is reflexive, symmetric, and transitive and so we can take linear mappings defined up to isomorphism as equivalence classes under this relation. Although we have loosened up how we associate vector spaces and linear mappings to spaces and continuous mappings, certain linear algebraic invariants remain meaningful, such as the dimension of equivalent vector spaces, and the rank of equivalent linear mappings.

With this notion of equivalence in mind, we establish the well-definedness of the proposed definitions. The central problem that needs resolution is the comparison of the homology of two simplicial complexes with the homeomorphic realizations. As a start, let's consider the relation between the homology of a space and its barycentric subdivision; by Theorem 10.12 we know that $|\mathrm{sd}\,K| = |K|$.

THEOREM 11.8. *There is an isomorphism of vector spaces $H_*(\mathrm{sd}\,K; \mathbb{F}_2) \cong H_*(K; \mathbb{F}_2)$.*

Proof: Recall the simplicial mapping $\lambda\colon \mathrm{sd}\,K \to K$, defined on vertices by "the last vertex,"

$$\lambda(\beta(S)) = \lambda(\beta(\{v_0, \ldots, v_q\})) = v_q.$$

This mapping is a simplicial approximation to the identity, $\mathrm{id}\colon |\mathrm{sd}\,K| \to |K|$. The simplicial mapping λ induces a linear mapping of chains $\lambda_*\colon C_*(\mathrm{sd}\,K; \mathbb{F}_2) \to C_*(K; \mathbb{F}_2)$.

To construct an inverse mapping to λ_*, we will not define another simplicial mapping, but work explicitly with the chains. Since we have explicit bases for the vector spaces of p-chains, it is possible to define linear mappings that do not necessarily come from a simplicial mapping. One such combinatorial mapping is defined for a fixed choice of vertex $b \in \mathrm{sd}\,K$, and generalizes the mapping i_{v_n} that figures in the computation of $H_p(\Delta^n[S]; \mathbb{F}_2)$.

Let $i_b\colon C_q(\mathrm{sd}\,K; \mathbb{F}_2) \to C_{q+1}(\mathrm{sd}\,K; \mathbb{F}_2)$ be given on the basis by

$$i_b(\{b_0, \ldots, b_q\}) = \begin{cases} \{b_0, \ldots, b_q, b\}, & \text{when } \{b_0, \ldots, b_q, b\} \text{ is nondegenerate in } \mathrm{sd}\,K, \\ \mathbf{0}, & \text{if } \{b_0, \ldots, b_q, b\} \text{ is degenerate or not in } \mathrm{sd}\,K. \end{cases}$$

The linear mapping i_b has the following properties:

$$\partial(i_b(S)) = S + i_b(\partial(S)), \qquad \text{and} \qquad \lambda_* \circ i_{\beta(S)} = i_{b_q} \circ \lambda_*, \text{ when } S = \{b_0, \ldots, b_q\}.$$

To prove these identities, we compute (where $\lambda(\beta(S_i)) = b_{\omega_i}$.)

$$\begin{aligned} \partial(i_b(S)) = \partial(\{b_0, \ldots, b_q, b\}) &= \{b_0, \ldots, b_q\} + \textstyle\sum_{i=0}^{q} \{b_0, \ldots, \widehat{b_i}, \ldots, b_q, b\} \\ &= S + i_b\left(\textstyle\sum_{i=0}^{q} \{b_0, \ldots, \widehat{b_i} \ldots, b_q\}\right) = S + i_b(\partial(S)). \\ \lambda_* \circ i_{\beta(S)}(\{\beta(S_0), \ldots, \beta(S_{q-1})\}) &= \lambda_*(\{\beta(S_0), \ldots, \beta(S_{q-1}), \beta(S)\}) \\ &= \{\lambda(\beta(S_0)), \ldots, \lambda(\beta(S_{q-1})), \lambda(\beta(S))\} \\ &= \{b_{\omega(0)}, \ldots, b_{\omega(q-1)}, b_q\} \\ &= i_{b_q}(\{b_{\omega(0)}, \ldots, b_{\omega(q-1)}\}) \\ &= i_{b_q} \circ \lambda_*(\{\beta(S_0), \ldots, \beta(S_{q-1})\}). \end{aligned}$$

Using these identities, we define the mapping $\beta_*\colon C_*(K; \mathbb{F}_2) \to C_*(\mathrm{sd}\,K; \mathbb{F}_2)$ by taking a simplex $S \in K$ to the sum of all the simplices in the barycentric subdivision of K that lie in $\Delta^q[S]$. Explicitly we can write

$$\beta_*(S) = \textstyle\sum_{S_0 \prec S_1 \prec \cdots \prec S_{q-1} \prec S} \{\beta(S_0), \beta(S_1), \ldots, \beta(S_{q-1}), \beta(S)\}.$$

However, this expression can be obtained more compactly by the recursive formula:

$$\beta_*(v) = v, \text{ if } v \text{ is a vertex in } K, \quad \beta_*(S) = i_{\beta(S)} \circ \beta_*(\partial(S)) \text{ if } \dim S > 0.$$

For example, $\beta_*(\{a,b\}) = i_{\beta(\{a,b\})}(\beta_*(a+b)) = \{a, \beta(\{a,b\})\} + \{b, \beta(\{a,b\})\}$, that is, the line segment ab is sent to the sum $am + bm$ where m is the midpoint of ab, the barycenter. We leave to the reader the induction argument that identifies the two descriptions of β_*.

In order that β_* defines a mapping on homology, we check the condition that $\partial \circ \beta_* = \beta_* \circ \partial$. On a 1-simplex, $\{a,b\}$, we have that

$$\partial(\beta_*(\{a,b\})) = \partial(\{a, \beta(\{a,b\})\} + \{b, \beta(\{a,b\})\}) = a + b = \beta_*(a+b) = \beta_*(\partial(\{a,b\})).$$

By induction on the dimension of a simplex, we have

$$\begin{aligned}\partial(\beta_*(S)) &= \partial(i_{\beta(S)}(\beta_*(\partial(S)))) = \beta_*(\partial(S)) + i_{\beta(S)}(\partial\beta_*(\partial(S))) \\ &= \beta_*(\partial(S)) + i_{\beta(S)}(\beta_*(\partial\partial(S))) = \beta_*(\partial(S)).\end{aligned}$$

Any linear mapping $m_*: C_p(K;\mathbb{F}_2) \to C_p(L;\mathbb{F}_2)$, defined for all p, that also satisfies $\partial \circ m_* = m_* \circ \partial$, is called a **chain mapping**; furthermore, a chain mapping m_* induces a linear mapping $m_*: H_p(K;\mathbb{F}_2) \to H_p(L;\mathbb{F}_2)$ for all p given by $m_*([v]) = [m_*(v)]$. We have showed that β_* is a chain mapping and so it induces a linear mapping for all p, $\beta_*: H_p(K;\mathbb{F}_2) \to H_p(\operatorname{sd} K;\mathbb{F}_2)$.

To finish the proof of the theorem, we show that β_* and $H(\lambda)$ are inverses. In one direction, we show that $\lambda_* \circ \beta_* = \mathrm{id}$ on $C_p(K;\mathbb{F}_2)$. On vertices $v \in K$, $\lambda_*(\beta_*(v)) = v$. By induction on dimension, we check on a p-simplex $S = \{v_0, \ldots, v_p\}$,

$$\lambda_*(\beta_*(S)) = \lambda_*(i_{\beta(S)}(\beta_*(\partial(S))) = i_{v_p}(\lambda_*(\beta_*(\partial(S)))) = i_{v_p}(\partial(S)) = S.$$

The last equation holds because $i_{v_p}(\partial(S)) = S + \partial(i_{v_p}(S))$, and $v_p \in S$ implies that $i_{v_p}(S) = \mathbf{0}$.

We next construct a chain homotopy $h: C_p(\operatorname{sd} K;\mathbb{F}_2) \to C_{p+1}(\operatorname{sd} K;\mathbb{F}_2)$ that satisfies

$$\partial \circ h + h \circ \partial = \beta_* \circ \lambda_* + \mathrm{id}.$$

This implies that $\beta_* \circ H(\lambda) = \mathrm{id}$ on $H_p(\operatorname{sd} K;\mathbb{F}_2)$ and establishes that β_* is the inverse of $H(\lambda)$. For $p = 0$, define $h(\beta(S)) = \{v_p, \beta(S)\}$, where $S = \{v_0, \ldots, v_p\}$. Since $\beta_*(\lambda_*(\beta(S))) = \beta_*(v_p) = v_p$, we have

$$\partial(h(\beta(S))) + h(\partial(\beta(S))) = \partial(\{v_p, \beta(S)\}) = v_p + \beta(S) = \beta_*(\lambda_*(\beta(S))) + \mathrm{id}(\beta(S)).$$

Note also that $h(\beta(S)) = \{v_p, \beta(S)\} \in C_1(\operatorname{sd} \Delta^p[S];\mathbb{F}_2) \subset C_1(\operatorname{sd} K;\mathbb{F}_2)$.

Suppose, by induction, that we have defined $h: C_k(\operatorname{sd} K;\mathbb{F}_2) \to C_{k+1}(\operatorname{sd} K;\mathbb{F}_2)$ for $k < p$. If $\{\beta(S_0), \ldots, \beta(S_k)\} \in C_k(\operatorname{sd} K;\mathbb{F}_2)$, then let $d_k = \dim(S_k)$. By induction, also assume that

$$h(\{\beta(S_0), \ldots, \beta(S_k)\}) \in C_{k+1}(\operatorname{sd} \Delta^{d_k}[S_k];\mathbb{F}_2) \subset C_{k+1}(\operatorname{sd} K;\mathbb{F}_2),$$

that is, the chains making up the value of h on a simplex in $\mathrm{sd}\,K$ lie in the subdivision of a particular simplex in K. Suppose T is a p-simplex and $T = \{\beta(S_0), \ldots, \beta(S_p)\}$ and $\dim(S_i) = d_i$. Consider the chain in $C_p(\mathrm{sd}\,K; \mathbb{F}_2)$ given by $\beta_*(\lambda_*(T)) + T + h(\partial(T))$. By induction, we can assume that $h(\partial(T)) \in C_p(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2)$ since the image under h of any $(p-1)$-simplex $\partial_i(T)$ in $\partial(T)$ lies in $C_{p-1}(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2) \oplus C_p(\mathrm{sd}\,\Delta^{d_{p-1}}[S_{p-1}]; \mathbb{F}_2) \subset C_p(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2)$. Since $S_0 \prec S_1 \prec \cdots \prec S_p$, we know that $T \in \mathrm{sd}\,\Delta^{d_p}[S_p]$. Finally, consider

$$\beta_*(\lambda_*(T)) = \beta_*(\{v_{\omega(0)}, \ldots, v_{\omega(p)}\}) \in C_p(\mathrm{sd}\,\Delta^p[\{v_{\omega(0)}, \ldots, v_{\omega(p)}\}]; \mathbb{F}_2)$$

Since $v_{\omega(i)}$ lies in $S_i \prec S_p$, we find $\beta_*(\lambda_*(T)) \in C_p(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2)$.

Putting these observations together it follows that the p-chain

$$\beta_*(\lambda_*(T)) + T + h(\partial(T)) \in C_p(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2).$$

The sequence of chains and boundary homomorphisms for $\mathrm{sd}\,\Delta^{d_p}[S_p]$ is exact in dimensions greater than zero because the operator $i_{\beta(S_p)}\colon C_k(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2) \to C_{k+1}(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2)$ satisfies $\partial \circ i_{\beta(S_p)} + \partial \circ i_{\beta(S_p)} = \mathrm{id}$ (the proof is the same as for $\Delta^{d_p}[S_p]$). Furthermore, by induction, we can assume that $\beta_* \circ \lambda_* + \mathrm{id} = h \circ \partial + \partial \circ h$ on $(p-1)$-chains, and so

$$\begin{aligned}
\partial(\beta_* \circ \lambda_* + \mathrm{id} + h \circ \partial) &= \partial \circ \beta_* \circ \lambda_* + \partial + (\partial \circ h) \circ \partial \\
&= \beta_* \circ \lambda_* \circ \partial + \partial + (\beta_* \circ \lambda_* + \mathrm{id} + h \circ \partial) \circ \partial \\
&= \beta_* \circ \lambda_* \circ \partial + \partial + \beta_* \circ \lambda_* \circ \partial + \partial + h \circ \partial \circ \partial = 0.
\end{aligned}$$

Thus

$$\beta_*(\lambda_*(T)) + T + h(\partial(T)) \in Z_p(\mathrm{sd}\,\Delta^{d_p}[S_p]) = B_p(\mathrm{sd}\,\Delta^{d_p}[S_p]).$$

Therefore, there is a $(p+1)$-chain $c_T \in C_{p+1}(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2) \subset C_{p+1}(\mathrm{sd}\,K; \mathbb{F}_2)$ with $\partial(c_T) = \beta_*(\lambda_*(T)) + T + h(\partial(T))$. Define $h(T) = c_T$. Carry out this construction for each $T \in K_p$ and extend linearly to define $h\colon C_p(\mathrm{sd}\,K; \mathbb{F}_2) \to C_{p+1}(\mathrm{sd}\,K; \mathbb{F}_2)$, satisfying $\beta_* \circ \lambda_* + \mathrm{id} = \partial \circ h + h \circ \partial$, and $h(T) \in C_{p+1}(\mathrm{sd}\,\Delta^{d_p}[S_p]; \mathbb{F}_2)$.

It now follows from Theorem 11.5 that $\beta_* \circ \lambda_*$ induces the identity on $H_p(\mathrm{sd}\,K; \mathbb{F}_2)$ and we have proved that $H_p(K; \mathbb{F}_2) \cong H_p(\mathrm{sd}\,K; \mathbb{F}_2)$, for all p. ◊

The trick of restricting and applying the exactness of the sequence of chains and boundary homomorphisms for a subcomplex of a simplicial complex is known generally as the *method of acyclic models*, introduced generally by S. Eilenberg (1913–1998) and J. Zilber in [Eilenberg-Zilber].

Since $|\mathrm{sd}\,K| = |K|$, Theorem 11.8 shows that subdivision does not change the homology up to isomorphism. The Simplicial Approximation Theorem, together with certain properties of simplicial mappings, will imply that the collection of homology vector spaces $\{H_p(K; \mathbb{F}_2) \mid p \geq 0\}$, are topological invariants.

Topological invariance of homology. *Suppose K and L are simplicial complexes with $|K|$ and $|L|$ homeomorphic. Then, for all p, the vector spaces $H_p(K; \mathbb{F}_2)$ and $H_p(L; \mathbb{F}_2)$ are isomorphic.*

Proof: Suppose $F\colon |K| \to |L|$ is a homeomorphism with inverse given by $G\colon |L| \to |K|$. Let $\phi\colon \mathrm{sd}^N K \to L$ be a simplicial approximation to F and $\gamma\colon \mathrm{sd}^M L \to K$ a simplicial approximation to G. Then, we can subdivide the simplicial mapping ϕ further to obtain $\mathrm{sd}^M \phi\colon \mathrm{sd}^{N+M} K \to \mathrm{sd}^M L$ which is also a simplicial approximation to F (Exercise 5, Chapter 10). The composite

$$\mathrm{sd}^{N+M} K \xrightarrow{\mathrm{sd}^M \phi} \mathrm{sd}^M L \xrightarrow{\gamma} K$$

is a simplicial approximation to the identity mapping $|\mathrm{sd}^{N+M} K| \to |K|$. Another approximation of the identity is given by the following composite:

$$\mathrm{sd}^{N+M} K \xrightarrow{\mathrm{sd}^{N+M-1}\lambda} \mathrm{sd}^{N+M-1} K \xrightarrow{\mathrm{sd}^{N+M-2}\lambda} \cdots \mathrm{sd}^2 K \xrightarrow{\mathrm{sd}\,\lambda} \mathrm{sd}\, K \xrightarrow{\lambda} K.$$

The proof of Theorem 11.8 shows that $H(\lambda)$ is an isomorphism between $H_p(\mathrm{sd}\, K; \mathbb{F}_2)$ and $H_p(K; \mathbb{F}_2)$ for all p. We next show that $H(\mathrm{sd}^j \lambda)$ is an isomorphism for all $j \geq 0$. More generally, consider the diagram of simplicial complexes and simplicial mappings:

$$\begin{tikzcd} \mathrm{sd}\, K \arrow[r, "\mathrm{sd}\,\eta"] \arrow[d, "\lambda"] & \mathrm{sd}\, L \arrow[d, "\lambda_K"] \\ K \arrow[r, "\eta"'] & L \end{tikzcd}$$

Here we define $\lambda_K\colon \mathrm{sd}\, L \to L$ as a simplicial approximation to the identity that satisfies $\lambda_K(\{\phi(v_0), \ldots, \phi(v_q)\}) = \phi(v_q)$, that is, we complete the diagram in such a way that $\eta \circ \lambda = \lambda_K \circ \mathrm{sd}\,\eta$. When we apply homology to these mappings, we obtain $H(\eta) \circ H(\lambda) = H(\lambda_K) \circ H(\mathrm{sd}\,\eta)$. Since λ and λ_K are simplicial approximations of the identity mapping, they are contiguous and so $H(\lambda_K)$ and $H(\lambda)$ are isomorphisms. Therefore, $H(\eta)$ and $H(\mathrm{sd}\,\eta)$ are equivalent as linear mappings of vector spaces. From this we deduce that $H(\mathrm{sd}^j \lambda)$ is an isomorphism for all $j \geq 0$.

Thus $\gamma \circ \mathrm{sd}^M \phi\colon \mathrm{sd}^{N+M} K \to K$ and $\lambda \circ (\mathrm{sd}\lambda) \circ \cdots \circ (\mathrm{sd}^{N+M-1}\lambda)\colon \mathrm{sd}^{N+M} K \to K$ are both simplicial approximations to the identity map $|\mathrm{sd}^{N+M} K| \to |K|$ and so they are contiguous by Lemma 10.19. Thus $H(\gamma) \circ H(\mathrm{sd}^M \phi) = H(\lambda) \circ H(\mathrm{sd}\lambda) \circ \cdots \circ H(\mathrm{sd}^{N+M-1}\lambda)$ which is an isomorphism. It follows that $H(\mathrm{sd}^M \phi)$ is one-one and also that $H(\phi)$ is one-one because it is equivalent to $H(\mathrm{sd}^M \phi)$.

By the same argument applied to $G \circ F = \mathrm{id}_{|L|}$, we form the composite

$$\mathrm{sd}^{N+M} L \xrightarrow{\mathrm{sd}^N \gamma} \mathrm{sd}^N K \xrightarrow{\phi} L$$

which is a simplicial approximation to $\mathrm{id}\colon |\mathrm{sd}^{N+M} L| \to |L|$ and so $H(\phi) \circ H(\mathrm{sd}^N \gamma)$ is an isomorphism and so $H(\phi)$ is onto. Thus we have proved that $H(\phi)\colon H_p(\mathrm{sd}^N K; \mathbb{F}_2) \to H_p(L; \mathbb{F}_2)$ is an isomorphism, for all p. By Theorem 11.8 and induction, $H_p(K; \mathbb{F}_2)$ is isomorphic to $H_p(\mathrm{sd}^N K; \mathbb{F}_2)$. Thus $H_p(K; \mathbb{F}_2) \cong H_p(L; \mathbb{F}_2)$ for all p. $\diamondsuit$

Corollary 11.9. *The Euler-Poincaré characteristic is a topological invariant of a triangulable space.*

Proof: Since $\chi(K)$ is calculable from the homology and homology is a topological invariant, we can write $\chi(K) = \chi(|K|)$ and compute the Euler-Poincaré characteristic from any triangulation of $|K|$. $\diamondsuit$

We can apply the corollary to prove a result known since the time of Euclid. A **Platonic solid** is a polyhedron with realization S^2 and for which all faces are congruent to a regular polygon, and each vertex has the same number of edges meeting there. Familiar examples are the tetrahedron and cube.

THEOREM 11.10. *There are only five Platonic solids.*

Proof: A polyhedron P need not be a simplicial complex, since the faces can be polygons not necessarily triangles (consider a soccer ball). However, if we subdivide each constituent polygon into triangles, we get a simplicial complex. The reader can now prove that the Euler-Poincaré characteristic $\chi(P)$, computed as the alternating sum $n_0 - n_1 + n_2$ where P has n_0 vertices, n_1 edges and n_2 faces, is the same for the subdivided polyhedron, a simplicial complex. Since P has realization S^2, we know that $\chi(P) = 2$.

Suppose each face has M edges (a regular M-gon) and, at each vertex, N faces meet. This leads to the relation:

$$M\, n_2/2 = n_1,$$

that is, each of the n_2 faces contributes M edges, but each edge is shared by two faces. It is also the case that

$$Nn_0/2 = n_1.$$

Since N faces meet at each vertex, N edges come into each vertex. But each edge has two vertices. Putting these relations into Euler's formula we get

$$\begin{aligned} 2 &= n_0 - n_1 + n_2 \\ &= (2n_1/N) - n_1 + (2n_1/M) \\ &= n_1((2/N) + (2/M) - 1). \end{aligned}$$

It follows that

$$\frac{n_1}{2} = \frac{MN}{2M + 2N - MN}.$$

If $N = 1$ or $N = 2$, there would be a boundary and so the polyhedron would fail to be a sphere. Since a Platonic solid encloses space, $N > 2$. Also $M \geq 3$ since each face is a polygon. Finally, n_1 must be an integer which is at least M.

These facts force $M < 6$. To see this, suppose $M \geq 6$ and $N > 2$. Then $2 - N < 0$ and we have

$$0 < 2M + 2N - MN = 2N + M(2 - N) \leq 2N + 6(2 - N) = 12 - 4N.$$

This implies that $4N < 12$, or that $N < 3$, which is impossible for N an integer and $N > 2$.

Setting $M = 3$ we get $n_1 = 6N/(6 - N)$ which is an integer when $N = 3$, 4, and 5. The case $N = 3$, $M = 3$ is realized by the tetrahedron; $N = 4$ and $M = 3$ is realized by the octahedron, and for $N = 5$, $M = 3$ by the icosahedron.

For $M = 4$ we have $n_1 = 8N/(8 - 2N) = 4N/(4 - N)$, and so $N = 3$ is the only case of interest which is realized by the cube. Finally, for $M = 5$ we have $n_1 = 10N/(10 - 3N)$ and so $N = 3$ is the only possible case, which gives the dodecahedron. $\diamondsuit$

Since the homology groups of a triangulable space are defined up to isomorphism, the invariants of vector spaces, like dimension, are topological invariants of the space. In the next result, we compare the dimension of one of the homology groups to a topological invariant introduced in Chapter 5.

THEOREM 11.11. *If K is a simplicial complex, then $\dim_{\mathbb{F}_2} H_0(K; \mathbb{F}_2) = \#\pi_0(|K|) =$ the number of path components of $|K|$.*

Proof: Consider the set K_0 of vertices of K. Define a relation on K_0 given by $v \sim v'$ if there is a 1-chain $c \in C_1(K; \mathbb{F}_2)$ with $\partial(c) = v + v'$. This relation is reflexive, because $\partial(\mathbf{0}) = v + v$; it is symmetric since $v + v' = v' + v$; and it is transitive because $\partial(c) = v + v'$ and $\partial(c') = v' + v''$ implies $\partial(c + c') = v + v' + v' + v'' = v + v''$. Let $[K_0]$ denote the set of equivalence classes under this relation. We show that $\#[K_0] = \dim_{\mathbb{F}_2} H_0(K; \mathbb{F}_2)$ and $\#[K_0] = \#\pi_0(|K|)$.

Consider the linear mapping $\mathbb{F}_2[[K_0]] \to H_0(K; \mathbb{F}_2)$ determined by $[v] \mapsto v + B_0(K)$. Since the equivalence relation is defined by the image of the boundary homomorphism, this mapping is well-defined. It is onto since every vertex in K lies in some equivalence class in $[K_0]$. We prove that this mapping is an isomorphism. Suppose that we make a choice of vertex in each equivalence class so that $[K_0] = \{[v_1], \ldots, [v_s]\}$. We show that the set of classes $\{v_i + B_0(K) \mid i = 1, \ldots, s\}$ is linearly independent in $H_0(K; \mathbb{F}_2)$. Suppose $v_{i_1} + \cdots + v_{i_r} + B_0(K) = B_0(K)$, that is, $v_{i_1} + \cdots + v_{i_r} = \partial(c)$ for some $c \in C_1(K; \mathbb{F}_2)$. We can write $c = e_1 + \cdots + e_t$ for edges $e_i \in K_1$. Since $v_{i_1} + \cdots + v_{i_r} = \partial(e_1 + \cdots + e_t)$ there is some edge, say e_1 with $\partial(e_1) = v_{i_1} + w_1$ for some vertex w_1. Since $v_{i_1} \sim w_1$, we know that $w_1 \neq v_{i_j}$ for $j = 2, \ldots, s$. It follows that we can replace v_{i_1} with w_1 and write

$$w_1 + v_{i_2} + \cdots + v_{i_r} = \partial(e_2 + \cdots + e_t).$$

By the same argument, we can choose e_2 with $\partial(e_2) = w_1 + w_2$. Once again, $w_1 \sim w_2$ and $w_2 \neq v_{i_j}$ for $j = 2, \ldots, s$. Therefore, $\partial(e_3 + \cdots + e_t) = w_2 + v_{i_2} + \cdots + v_{i_r}$. Continuing in this manner, we get down to $\partial(e_t) = w_{t-1} + v_{i_2} + \cdots + v_{i_r}$, which is impossible since the vertices v_{i_j} and w_{t-1} are not equivalent under the relation. Thus $\#[K_0] = s = \dim_{\mathbb{F}_2} H_0(K; \mathbb{F}_2)$.

To finish the proof, we show that $\#[K_0] = \#\pi_0(|K|)$. First notice that the open star of a vertex, $O_K(v)$ is path-connected. This follows because there is a path joining the vertex v to every point in $O_K(v)$. Recall that the set of path components, $\pi_0(|K|)$ is the set of equivalence classes of points in $|K|$ under the relation that two points are equivalent if there is a path in $|K|$ joining them. Denote the equivalence classes under this relation by $\langle x \rangle$. Suppose $[v_i] \in [K_0]$ is a class of vertices under the relation $v_i \sim w$ if there is a 1-chain c with $\partial(c) = v_i + w$. Let $U_i = \bigcup_{w \in [v_i]} O_K(w)$. We show that U_i is a path component of $|K|$ and that $U_i \cap U_j = \emptyset$ when $i \neq j$. Notice that U_i is path connected—we only need to show that the vertices are joined by paths since each $O_K(w)$ is path connected. By w and w' satisfy $w + w' = \partial(c)$ and the 1-chain c determines a path joining w and w'. Furthermore, if there is a path joining v_i to a point x in $|K|$, then there is a path joining v_i to some vertex v in K, and the path joining v_i to v can be deformed to pass only along edges of K, whose sum gives a 1-chain c with $\partial(c) = v_i + v$, that is, $v \in U_i$ and $U_i = \langle v_i \rangle$. Suppose $x \in U_i \cap U_j$. Then there are vertices w and v with $v \sim v_i$ and $w \sim v_j$ and $x \in O_K(v) \cap O_K(w)$. However, this implies that $x \in \Delta^m[S]$ for some m-simplex S in K for which $v, w \in S$. This implies that $e = \{v, w\} \prec S$ is an edge with $\partial(e) = v + w$ and so $v \sim w$ which implies $v_i \sim v_j$, a contradiction. Thus $|K|$ is partitioned into disjoint path components $\langle v_1 \rangle = U_1, \ldots, \langle v_s \rangle = U_s$. ♢

We return to the central question of the book.

Invariance of dimension for (m, n): *If $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$, then $n = m$.*

Proof: We make this a question about simplicial complexes by using the one-point compactification (Definition 6.11). If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then their one-point compactifications are homeomorphic. Since $\mathbb{R}^l \cup \{\infty\}$ is homeomorphic to S^l, it follows that $\mathbb{R}^n \cong \mathbb{R}^m$ implies $S^n \cong S^m$.

By the topological invariance of homology, and the homeomorphism $S^n \cong |\mathrm{bdy}\,\mathbf{\Delta^{n+1}}|$, we have

$$H_p(S^n; \mathbb{F}_2) \cong H_p(\mathrm{bdy}\,\mathbf{\Delta^{n+1}}; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & p = 0, n, \\ \{\mathbf{0}\} & \text{else.} \end{cases}$$

If $S^n \cong S^m$, then $H_p(S^n; \mathbb{F}_2) \cong H_p(S^m; \mathbb{F}_2)$ for all p and, by our computation of the homology of spheres, this is only possible if $n = m$. ♢

The first proofs of this theorem were due to Brouwer [Brouwer] and Lebesgue [Lebesgue]. Brouwer's proof was based on simplicial approximation and used an index, defined generically as the cardinality of the preimage of a point, to obtain a contradiction to the existence of a homeomorphism between $[0,1]^n = [0,1] \times \cdots \times [0,1]$ (n times) and $[0,1]^m$ when $n \neq m$. Lebesgue's first proof was not rigorous, but introduced a point-set definition of dimension that led to the modern development of the subject of dimension theory. An account of these developments can be found in [Johnson] and [Hurewicz-Wallman].

Another famous theorem of Brouwer can be proved using homology, generalizing the argument in Theorem 8.7 in which the fundamental group of S^1 played a key role.

The Brouwer fixed point theorem. *If $e^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$ denotes the* n**-disk** *and $f\colon e^n \to e^n$ is a continuous mapping, then there is a point $\mathbf{x}_0 \in e^n$ with $f(\mathbf{x}_0) = \mathbf{x}_0$, that is, e^n has the fixed point property.*

Proof: Suppose that $f\colon e^n \to e^n$ is a continuous mapping without fixed points. If $\mathbf{y} \in e^n$, then $\mathbf{y} \neq f(\mathbf{y})$. Join $f(\mathbf{y})$ to $\mathbf{y}$ and continue this ray until it meets $S^{n-1} = \mathrm{bdy}\, e^n$ and denote this point by $g(\mathbf{y})$. We can characterize $g(\mathbf{y})$ by $g(\mathbf{y}) = (1-t)f(\mathbf{y}) + t\mathbf{y}$ where $t > 0$ and $\|g(\mathbf{y})\| = 1$. Because we are in a nicely behaved inner product space, the argument for the case of $n = 2$ (Theorem 8.7) carries over exactly to prove that $g\colon e^n \to S^{n-1}$ is continuous. Furthermore, by the definition of g, $g \circ i\colon S^{n-1} \to S^{n-1}$ is the identity when $i\colon S^{n-1} \to e^n$ is the inclusion of the boundary.

Apply homology to this composite $\mathrm{id}_{S^{n-1}} = g \circ i$ to obtain $H(\mathrm{id}_{S^{n-1}})$, an isomorphism, written as $H(g) \circ H(i)$. However, $H_{n-1}(S^{n-1}; \mathbb{F}_2) \neq \{\mathbf{0}\}$ while $H_{n-1}(e^n; \mathbb{F}_2) = \{\mathbf{0}\}$, because e^n is homeomorphic to $\mathbf{\Delta}^n$. Thus, $H(i)\colon H_{n-1}(S^{n-1}; \mathbb{F}_2) \to H_{n-1}(e^n; \mathbb{F}_2)$ is the zero homomorphism $[c] \mapsto \mathbf{0}$. An isomorphism $H(\mathrm{id}_{S^{n-1}})\colon H_{n-1}(S^{n-1}; \mathbb{F}_2) \to H_{n-1}(S^{n-1}; \mathbb{F}_2)$

cannot be factored as $H(g) \circ ([c] \mapsto \mathbf{0})$, and so a continuous mapping $f\colon e^n \to e^n$ without fixed points cannot exist. $\diamondsuit$

The Brouwer fixed point theorem was a significant signpost in the development of topology. The theory of fixed points of mappings plays an important role throughout mathematics and its applications. With more refined notions of homology, deep generalizations of the Brouwer fixed point theorem can be proved. See [Munkres2] for examples, like the Lefschetz-Hopf fixed point theorem.

In dimension two we proved a case of the Borsuk-Ulam theorem (Theorem 8.10)—there does not exist a continuous function $f\colon S^2 \to S^1$ with $f(-\mathbf{x}) = -f(\mathbf{x})$ for all $\mathbf{x} \in S^2$. The higher dimensional version of the Borsuk-Ulam theorem treats mappings $f\colon S^n \to S^{n-1}$ for which $f(-\mathbf{x}) = -f(\mathbf{x})$. The general setting for this discussion involves the notion of a space with involution.

Definition 11.12. *A space X has an* **involution** $\nu\colon X \to X$ *if ν is continuous and $\nu \circ \nu = \mathrm{id}_X$. If (X, ν) and (Y, μ) are spaces with involution, then an* **equivariant mapping** *$g\colon X \to Y$ is a continuous mapping satisfying $g \circ \nu = \mu \circ g$.*

Consider the antipodal mapping on S^n and on S^{n-1} given by $a(\mathbf{x}) = -\mathbf{x}$. The general Borsuk-Ulam theorem states that a continuous mapping $f\colon S^n \to S^{n-1}$ cannot be equivariant, that is, $f(a(\mathbf{x})) = a(f(\mathbf{x}))$ does not hold for all $\mathbf{x} \in S^n$.

Assuming this formulation of the Borsuk-Ulam theorem, we observe an immediate consequence: If we let $F\colon S^n \to \mathbb{R}^n$ be any continuous mapping that satisfies $F(\mathbf{x}) \neq F(-\mathbf{x})$ for all $\mathbf{x} \in S^n$, we can define

$$g(\mathbf{x}) = \frac{F(\mathbf{x}) - F(-\mathbf{x})}{\|F(\mathbf{x}) - F(-\mathbf{x})\|}.$$

Then $g\colon (S^n, a) \to (S^{n-1}, a)$ is an equivariant mapping. By the Borsuk-Ulam Theorem, no such mapping exists, and so there must be a point $\mathbf{x}_0 \in S^n$ with $F(\mathbf{x}_0) = F(-\mathbf{x}_0)$, that is, two antipodal points are mapped to the same point. It follows from this that no subspace of $\mathbb{R}^n$ is homeomorphic to S^n.

We deduce the Borsuk-Ulam theorem as a corollary of a theorem of Walker [Walker] which deals with the homology of equivariant mappings. Assume that (X, ν) is a space with involution and that X is triangulable. Then there is a simplicial complex K with $|K| \cong X$ and a simplicial mapping $\bar{\nu}\colon K \to K$ with $|\bar{\nu}| \simeq \nu$ and $\bar{\nu} \circ \bar{\nu} = \mathrm{id}_K$. An argument for the existence of K and $\bar{\nu}$ can be made using simplicial approximation. For the sphere, we can do even better. For example, one triangulation of S^2 is the octahedron on which we can write down an explicit simplicial mapping which realizes the antipodal map. Higher dimensional models of this sort exist for every sphere. Note that the antipodal mapping on the sphere has no fixed points. We will assume that a simplicial approximation to the antipodal map can be chosen without fixed points as well, and so any simplex S in L satisfies $\bar{a}(S) \cap S = \emptyset$ where $\bar{a}\colon L \to L$ realizes the antipode on $|L| \cong S^n$.

Theorem 11.13. *If (X, ν) is a triangulable space with involution and $F\colon (X, \nu) \to (S^n, a)$ is an equivariant mapping, then there is a homology class $[c] \in H_j(X; \mathbb{F}_2)$ with $1 \leq j \leq n$, $[c] \neq \mathbf{0}$ and $H(\nu)([c]) = [c]$. Furthermore, if the least dimension in which this condition holds is $j = n$, then the class $[c]$ can be chosen such that $H(F)([c]) = [u] \neq \mathbf{0}$ in $H_n(S^n; \mathbb{F}_2)$.*

Proof: Let us assume that we have triangulations for (X, ν) and (S^n, a) denoted by $(K, \bar{\nu})$ and $(L, \bar{a})$. Let $\phi: K \to L$ be a simplicial equivariant mapping with ϕ a simplicial approximation to F. Let $\theta_K = \mathrm{id}_{K*} + \bar{\nu}_*: C_j(K; \mathbb{F}_2) \to C_j(K; \mathbb{F}_2)$ and $\theta_L = \mathrm{id}_{L*} + \bar{a}_*: C_j(L; \mathbb{F}_2) \to C_j(L; \mathbb{F}_2)$. Since $\bar{\nu}$ and $\bar{a}$ are simplicial mappings, $\theta_K \circ \partial = \partial \circ \theta_K$ and likewise for θ_L. Also $\theta_K \circ \theta_K = \mathbf{0}$, because

$$(\mathrm{id}_{K*} + \bar{\nu}_*) \circ (\mathrm{id}_{K*} + \bar{\nu}_*) = \mathrm{id}_{K*} + \bar{\nu}_* + \bar{\nu}_* + (\bar{\nu} \circ \bar{\nu})_* = 2\mathrm{id}_{K*} + 2\bar{\nu}_* = \mathbf{0},$$

and similarly, $\theta_L \circ \theta_L = \mathbf{0}$.

If there is a class $\mathbf{0} \neq [c] \in H_j(K; \mathbb{F}_2)$ with $H(\bar{\nu})([c]) = [c]$ and $0 < j < n$, then we are done. So, let us assume that if $H(\bar{\nu})([c]) = [c]$, then $[c] = \mathbf{0}$. Notice that $H(\bar{\nu})([c]) = [c]$ if and only if $[\theta_K(c)] = \mathbf{0}$.

Let $h_0 \in L$ denote a vertex. The homology class $[h_0] = h_0 + B_0(L) \in H_0(L; \mathbb{F}_2)$ satisfies $[\theta_L(h_0)] = \mathbf{0}$, since $H_0(L; \mathbb{F}_2)$ has dimension one, and both id_L and $\bar{a}$ induce the identity on $H_0(L; \mathbb{F}_2)$. It follows that there is a 1-chain h_1 with $\partial(h_1) = \theta_L(h_0)$. Notice that

$$\partial(\theta_L(h_1)) = \theta_L(\partial(h_1)) = \theta_L(\theta_L(h_0)) = \mathbf{0}.$$

Since $|L| \cong S^n$, $B_1(L) = Z_1(L)$ and so $\theta_L(h_1) = \partial(h_2)$ for some $h_2 \in C_2(L; \mathbb{F}_2)$. It is also the case that $\theta_L(h_1) \neq \mathbf{0}$. To see this, suppose $h_1 = e_1 + e_2 + \cdots + e_t$. Then we can number the edges e_i with $\partial(e_1) = h_0 + v_1$, $\partial(e_i) = v_{i-1} + v_i$ and $\partial(e_t) = v_{t-1} + \bar{a}_*(h_0)$. If $\theta_L(h_1) = \mathbf{0}$, then we deduce $\bar{a}_*(e_i) = e_{t-i+1}$ from which we find either an edge that is its own antipode, or a pair of edges sharing antipodal vertices. By the assumption that the antipode $\bar{a}$ has no fixed points, we find $\theta_L(h_1) \neq \mathbf{0}$.

We repeat this construction to find $h_j \in C_j(L; \mathbb{F}_2)$, for $1 \leq j \leq n$, with $\partial(h_j) = \theta_L(h_{j-1})$. By the same argument showing $\theta_L(h_1) \neq \mathbf{0}$, we find $\theta_L(h_j) \neq \mathbf{0}$ for $1 \leq j \leq n$. Consider $\theta_L(h_n)$; since $\theta_L(h_n) \neq \mathbf{0}$, $[\theta_L(h_n)]$ generates $H_n(L; \mathbb{F}_2)$. The chains h_j may be thought of as generalized hemispheres.

We have assumed that, if $1 \leq j < n$, and $[c] \in H_j(K; \mathbb{F}_2)$ satisfies $H(\bar{\nu})[c] = [c]$, then $[c] = \mathbf{0}$. We use this to make an analogous construction of classes $c_j \in C_j(K; \mathbb{F}_2)$ with properties like the h_j. Let $c_0 \in K$ be a vertex. Then $[\theta_K(c_0)] = \mathbf{0}$, and so there is a 1-chain c_1 with $\partial(c_1) = \theta_K(c_0)$. The 1-chain $\theta_K(c_1)$ satisfies

$$\partial(\theta_K(c_1)) = \theta_K(\partial(c_1)) = \theta_K(\theta_K(c_0)) = \mathbf{0}.$$

Thus $\theta_K(c_1)$ is a 1-cycle. However, $\theta_K(\theta_K(c_1)) = \mathbf{0}$, so $\theta_K(c_1) = \partial(c_2)$ for some 2-chain c_2. Continuing in this manner, we find chains c_j satisfying $\partial(c_j) = \theta_K(c_{j-1})$ for $1 \leq j \leq n$.

We next define another sequence of chains on L. We know that $h_0 + \phi_*(c_0)$ is a 0-cycle, and so there is a chain u_1 with $\partial(u_1) = h_0 + \phi_*(c_0)$. Consider $h_1 + \phi_*(c_1) + \theta_L(u_1)$. Then

$$\begin{aligned}\partial(h_1 + \phi_*(c_1) + \theta_L(u_1)) &= \partial(h_1) + \phi_*(\partial(c_1)) + \theta_L(\partial(u_1)) \\ &= \theta_L(h_0) + \phi_*(\theta_K(c_0)) + \theta_L(h_0 + \phi_*(c_0)) \\ &= \theta_L(h_0) + \theta_L(\phi_*(c_0)) + \theta_L(h_0) + \theta_L(\phi_*(c_0)) = \mathbf{0}.\end{aligned}$$

Here we have used $\theta_L \circ \phi_* = \phi_* \circ \theta_K$ which holds by the assumption that ϕ is equivariant. It follows that there is a 2-chain u_2 with $\partial(u_2) = h_1 + \phi_*(c_1) + \theta_L(u_1)$. The analogous computation shows $h_2+\phi_*(c_2)+\theta_L(u_2)$ is a cycle and so there is a 3-chain with $\partial(u_3) = h_2+\phi_*(c_2)+\theta_L(u_2)$. Continuing in this manner, we find j-chains u_j with $\partial(u_j) = h_{j-1}+\phi_*(c_{j-1})+\theta_L(u_{j-1})$ for $1 \leq j \leq n$ ($u_0 = \mathbf{0}$). By construction, $h_n+\phi_*(c_n)+\theta_L(u_n)$ is an n-cycle in $C_n(L;\mathbb{F}_2)$ and so it is homologous to either $\theta_L(h_n)$ or to $\mathbf{0}$ since $H_n(L;\mathbb{F}_2) \cong \mathbb{F}_2[\{[\theta_L(h_n)]\}]$. In either case, $\theta_L(h_n+\phi_*(c_n)+\theta_L(u_n)) = \theta_L(h_n)+\phi_*(\theta_K(c_n))$ is homologous to $\mathbf{0}$. Let $c = \theta_K(c_n)$, then $\partial(c) = \partial(\theta_K(c_n)) = \theta_K(\partial(c_n)) = \theta_K(\theta_K(c_{n-1})) = \mathbf{0}$, and so $[c] \in H_n(K;\mathbb{F}_2)$ satisfies $H(\phi)([c]) = [\phi_*(c)] = [\phi_*(\theta_K(c_n))] = [\theta_L(h_n)]$ and $[\bar{\nu}_*(c)] = [\bar{\nu}_*(\theta_K(c_n))] = [\theta_K(c_n)] = [c]$, so $H(\nu)([c]) = [c]$. $\diamondsuit$

Corollary 11.14. *There are no equivariant mappings $F\colon (S^n, a) \to (S^m, a)$ when $n > m$.*

Proof: The homology of S^n has no nonzero classes in $H_j(S^n;\mathbb{F}_2)$ for $1 \leq j \leq m$, and so, if there were an equivariant mapping $F\colon S^n \to S^m$, the conclusion of Theorem 11.13 would fail . $\diamondsuit$

The Borsuk-Ulam theorem is the case $m = n-1$. There are many proofs of the Borsuk-Ulam theorem, as well as remarkable applications in diverse parts of mathematics. The interested reader should consult [Matoušek] for more details (and a great read).

Exercises

1. Suppose X and Y are triangulable space that are homotopy equivalent. Show that $H_p(X;\mathbb{F}_2) \cong H_p(Y;\mathbb{F}_2)$ for all p. The notion of contiguous simplicial mappings (Theorem 10.21) plays a big role here.

2. Use the homotopy invariance of homology to compute the homology of the Möbius band.

3. The projective plane, $\mathbb{R}\mathrm{P}^2$ is modeled by an explicit simplicial complex, as shown in Chapter 10. The combinatorial data allow one to construct the sequence of boundary homomorphisms

$$C_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \xrightarrow{\partial} C_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \xrightarrow{\partial} C_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \to \{\mathbf{0}\}.$$

   This may be boiled down to a pair of matrices whose ranks determine the homology. Use this formulation to compute $H_j(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ for all j.

4. If L is a subcomplex of a simplicial complex K, $L \subset K$, then we can define the homology of the pair (K, L) by setting

$$C_p(K, L;\mathbb{F}_2) = C_p(K;\mathbb{F}_2)/C_p(L;\mathbb{F}_2).$$

   Show that the boundary operator on the chains on K and L defines a boundary operator on the quotient vector space $C_p(K, L;\mathbb{F}_2)$. Then $H_p(K, L;\mathbb{F}_2)$ is the quotient of the kernel of the boundary operator by the image of the boundary operator. Compute

$H_p(K, L; \mathbb{F}_2)$ for all p when K is a cylinder $S^1 \times [0,1]$ and L is its boundary (a pair of circles), and when K is the Möbius band, and L its boundary.

5. A path through a simplex can be deformed to pass only through the subcomplex of edges (1-simplices) of the simplex. Because a simplex is convex, this gives a homotopy between the path and its deformation. Use this idea to define a mapping $\pi_1(|K|, v_0) \to H_1(K; \mathbb{F}_2)$ that sends a loop based at a vertex v_0 to a 1-chain in K. Show that the mapping so defined is a group homomorphism. What happens in the case that $|K| \cong S^1$?

Where from here?

The diligent reader who has mastered the better part of this book is ready for a great deal more. I have restricted my attention to particular spaces and particular methods in order to focus on the question of the topological invariance of dimension. The quick route to the proof of invariance of dimension left a lot of the landscape unexplored. In particular, the question of dimension can be posed more generally, for which a rich theory has been developed. The interested reader can consult [Hurewicz-Wallman] for the classic treatment, and the articles of Johnson [Johnson], and Dauben [Dauben] for a history of its development. For topics in the general history of topology, there is the collection of essays edited by James [James] and the sweeping account of Dieudonné [Dieudonne].

Where to go next is best answered by recommending some texts for which the reader is now ready.

A far broader treatment of the topics in this book can be found in the books of Munkres, [Munkres1] and [Munkres2]. Enthusiasts of point-set topology (Chapters 1–6) will find a rich vein there. Other treatments of point-set topics can be found in [Kahn] and [Henle], and there is the collection of sometimes surprising counterexamples to sharpen point-set topological intuition found in [Steen-Seebach].

The fundamental group is thoroughly presented in the classic book of Massey [Massey] and in the lectures of Lima [Lima]. A deeper exploration of the idea of covering spaces leads to a topological setting for a Galois correspondence, which has been a fruitful analogy.

For the purposes of ease of exposition toward our main goal, I introduced homology with coefficients in $\mathbb{F}_2$. It is possible to define homology with other coefficients, $H_*(X; A)$ for A an abelian group, and for arbitrary topological spaces, singular homology, by developing the properties of simplices with more care. This is the usual place to start a graduate course in algebraic topology. I recommend [Massey], [Munkres2], [Greenberg-Harper], [Hatcher], [Spanier] and [Crossley] for these topics. With more subtle chains, many interesting geometric results can be proved.

The most important examples of topological spaces throughout the history of topology are manifolds. These are spaces which are locally homeomorphic to open sets in $\mathbb{R}^n$ for which the methods of the Calculus play a principal role. The interface between topology and analysis is subtle and made clear on manifolds. This is the subject of differential topology, treated in [Milnor], [Dubrovin-Fomenko-Novikov], and [Madsen-Tornehave].

I did not treat some of the other classical topological topics in this book about which the reader may curious. On the subject of knots, the books of Colin Adams [Adams] and Livingston [Livingston] are good introductions. The problem of classifying all surfaces is presented in [Massey] and [Armstrong]. Geometric topics, like the Poincaré index theorem, are a part of classical topology, and can be read about in [Henle].

Finally, the notation $\pi_0(X)$ and $\pi_1(X)$ hints at a sequence of groups, $\pi_n(X)$, known as the higher homotopy groups of a space X. The iterative definition, introduced by Hurewicz [Hurewicz], is

$$\pi_n(X) = \pi_{n-1}(\Omega(X, x_0)).$$

For example, the second homotopy group of X is the fundamental group of the based loop space on X. The properties of these groups and their computation for particular spaces X is a difficult problem. Some aspects of this problem are developed in [Croom], [Maunder], [May], and [Spanier].

To the budding topologist, I wish many exciting discoveries.